Holographic Rényi entropy for two-dimensional $\mathcal{N}$=(2,2) superconformal field theory

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Abstract

We investigate the holographic Rényi entropy for two-dimensional $\mathcal{N} = (2,2)$ superconformal field theory (SCFT), which is dual to $\mathcal{N} = 2$ supergravity in AdS_3 background. In SCFT we have the stress tensor, current, and their supersymmetric partners, and in supergravity we have the graviton, vector field, and two gravitinos. We get the Rényi mutual information of two short intervals on complex plane in expansion by the cross ratio x to order x^4, and Rényi entropy of one interval on torus in expansion by $q = \exp(-2\pi\beta/L)$, with β being the inverse temperature and L being the spatial period, to order q^2. We calculate in both the supergravity and SCFT sides, and find matches of the results.

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Contents

1 Introduction

Holographic entanglement entropy [1, 2] is an interesting subject in the investigation of AdS/CFT (anti-de Sitter/conformal field theory) correspondence [3–6]. To calculate the entanglement entropy one can use the replica trick, calculate the n-th Rényi entropy with $n \in \mathrm{Z}$ and $n > 1$, and take the $n \to 1$ limit to get the entanglement entropy [7–9]. Rényi entropy is not only a tool to calculate the entanglement entropy but also interesting in its own right, and one can directly investigate the holographic Rényi entropy [10–16].

Quantum gravity in AdS_3 background with cosmological constant $\Lambda = -\frac{1}{R^2}$ and small Newton constant G_N is dual to a two-dimensional CFT with a large central charge [17]

$$c = \frac{3R}{2G_N}. \tag{1.1}$$

There are many investigations of holographic Rényi entropy in $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [10, 13–15, 18–34]. In gravity side, one can not only get the classical part [14, 15, 25] but also use the partition function of Einstein gravity in handlebody background [35–37] and get the one-loop corrections [15]. In CFT side one can use the operator product expansion (OPE) of twist operators to calculate the Rényi mutual information of two short intervals on complex plane [10, 13, 18, 19], and use the low temperature expansion of density matrix to calculate the Rényi entropy of one interval on torus with low temperature [24, 25].

In this paper we investigate the holographic Rényi entropy in another kind of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the dual between the $\mathcal{N} = 2$ supergravity in AdS_3 background and the $\mathcal{N} = (2, 2)$ superconformal field theory (SCFT). In supergravity side we have the fields of graviton, vector and two gravitinos, and we denote them by g, v and ψ, respectively. Correspondingly, in SCFT side we have the stress tensor T, $\bar{T}$, current J, $\bar{J}$, and their

superpartners G, $\bar{G}$, H, $\bar{H}$. Note that the duality between the bosonic part of the supergravity with only graviton and vector field and the bosonic part of the SCFT with only stress tensor and current is self-consistent, and we will also consider the holographic Rényi entropy in such a correspondence.

The remaining part of this paper is arranged as follows. In section 2, we investigate the Rényi mutual information of two short intervals on complex plane. In section 3, we consider the Rényi entropy of one interval on tours with low temperature. Then there is the conclusion and discussion in section 4. In appendix A, we review some details of the $\mathcal{N}=(2,2)$ SCFT. In appendix B, we give some useful summation formulas.

2 Rényi mutual information of two intervals on complex plane

In this section we calculate the Rényi mutual information of two short intervals on complex plane. In other words, we have the ground state SCFT on space of one-dimensional infinite straight line. We expand the Rényi mutual information by the small cross ratio x to order x^4. In gravity side we use method in [15], and in CFT side we use OPE of twist operators in [10, 18, 19].

2.1 Supergravity result

The classical part of the holographic Rényi mutual information only depends on the classical configuration of the gravity background, and so it is the same as that of pure Einstein gravity. The result can be found in [15]

$$I_n^{\mathrm{cl}}=\frac{c(n-1)(n+1)^2x^2}{144n^3}+\frac{c(n-1)(n+1)^2x^3}{144n^3}+\frac{c(n-1)(n+1)^2(1309n^4-2n^2-11)x^4}{207360n^7}+O(x^5). \tag{2.1}$$

To calculate the one-loop part of the holographic Rényi mutual information to order x^4, we need the contributions of the consecutively decreasing words (CDWs) and the 2-CDWs [15]. The one-loop part depends on the field content of the gravity theory. For the graviton g we have [15]

$$I_{n,g}^{\text{1-loop}}=\frac{(n+1)(n^2+11)(3n^4+10n^2+227)x^4}{3628800n^7}+O(x^5), \tag{2.2}$$

for the vector field v we have

$$\begin{aligned}I_{n,v}^{\text{1-loop}}=&\frac{(n+1)(n^2+11)x^2}{720n^3}+\frac{(n+1)(23n^4+233n^2-40)x^3}{15120n^5}\\&+\frac{(n+1)(2129n^6+20469n^4-6285n^2+2695)x^4}{1451520n^7}+O(x^5),\end{aligned} \tag{2.3}$$

and for the two gravitinos ψ we have

$$I_{n,\psi}^{\text{1-loop}}=\frac{(n+1)(2n^4+23n^2+191)x^3}{30240n^5}+\frac{(n+1)(33n^6+358n^4+2857n^2-368)x^4}{302400n^7}+O(x^5). \tag{2.4}$$

2.2 Contributions of T in SCFT

Using the replica trick we get the n-fold of the original SCFT, which we call SCFT^n. In SCFT^n we use OPE of twist operators to calculate the partition function, and the Rényi mutual information turns out to be [19–21]

$$I_n=\frac{2}{n-1}\log\Big[\sum_K\alpha_K d_K^2x^{h_K}{}_2F_1(h_K,h_K;2h_K;x)\Big], \tag{2.5}$$

where the summation of K is over all the linearly independent orthogonalized holomorphic quasiprimary operators Φ_K in SCFT^n. Here α_K, d_K, h_K are, respectively, the normalization factor, OPE coefficient, and conformal

level	operator	degeneracy	number
0	1	1	1
2	T	n	n
4	$\mathcal{A}$	n	$\frac{n(n+1)}{2}$
	TT	$\frac{n(n-1)}{2}$	

Table 1: To level 4 the holomorphic quasiprimary operators in SCFT^n with contributions of only T. We follow the convention in [31] and omit the replica indices that take values from 0 to $n-1$. Note that the 4th column agrees with the counting in (2.6).

weight of Φ_K. Note that there is a factor 2 in the right-hand side of (2.5), and this concerns the contributions from the anti-holomorphic sector.

In this subsection we repeat the calculation in [19], with the contributions of only the vacuum conformal family, i.e. the contributions of operators that are constructed solely by T. To level 4 the SCFT^n quasiprimary operators we have to consider are counted as

$$(1-x)\chi_{(0)}^n + x = 1 + nx^2 + \frac{n(n+1)}{2}x^4 + O(x^5), \tag{2.6}$$

with $\chi_{(0)}$ in (A.1), and they are listed in table 1. For these quasiprimary operators we have the normalization factors

$$\alpha_1 = 1, \quad \alpha_T = \frac{c}{2}, \quad \alpha_{\mathcal{A}} = \frac{c(5c+22)}{10}, \quad \alpha_{TT} = \frac{c^2}{4}, \tag{2.7}$$

and the OPE coefficients

$$d_1 = 1, \quad d_T = \frac{n^2-1}{12n^2}, \quad d_{\mathcal{A}} = \frac{(n^2-1)^2}{288n^4}, \quad d_{TT}^{j_1j_2} = \frac{1}{8n^4c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^2}{144n^4}. \tag{2.8}$$

Using (2.5) we get the Rényi mutual information organized by the large central charge

$$I_n^{(0)} = I_n^{\mathrm{L}} + I_{n,(0)}^{\mathrm{NL}} + \cdots, \tag{2.9}$$

with the leading part I_n^{L} equaling I_n^{cl} (2.1) and the next-to-leading part $I_{n,(0)}^{\mathrm{NL}}$ equaling $I_{n,g}^{\text{1-loop}}$ (2.2).

2.3 Contributions of T, J in SCFT

By adding the contributions of J, we have to consider the extra quasiprimary operators that are counted as

$$(1-x)\chi_{(0)}^n(\chi_{(1)}^n - 1) = nx + \frac{n(n+1)}{2}x^2 + \frac{n(n^2+12n-1)}{6}x^3 + \frac{n(n^3+26n^2+59n+10)}{24}x^4 + O(x^5), \tag{2.10}$$

with $\chi_{(0)}$ being (A.1) and $\chi_{(1)}$ being (A.6), and they are listed in table 2. Only a few of them contribute to the Rényi mutual information. For the relevant operators we have the normalization factors

$$\begin{aligned}
&\alpha_{JJ} = \frac{c^2}{9}, \quad \alpha_{I(JJ)} = \frac{4c^2}{9}, \quad \alpha_{J\mathcal{K}} = \frac{c^2(c+2)}{18}, \quad \alpha_{\mathcal{M}\mathcal{M}} = \frac{4c^2(c-1)^2}{81}, \\
&\alpha_{TJJ} = \frac{c^3}{18}, \quad \alpha_{JJ\mathcal{M}} = \frac{2c^3(c-1)}{81}, \quad \alpha_{JJJJ} = \frac{c^4}{81}, \quad \alpha_{II(JJ)} = \frac{20c^2}{27},
\end{aligned} \tag{2.11}$$

and the OPE coefficients

$$\begin{aligned}
&d_{JJ}^{j_1j_2} = -\frac{3}{4n^2c}\frac{1}{s_{j_1j_2}^2}, \quad d_{I(JJ)}^{j_1j_2} = -\frac{3}{8n^3c}\frac{c_{j_1j_2}}{s_{j_1j_2}^3}, \quad d_{J\mathcal{K}}^{j_1j_2} = -\frac{n^2-1}{16n^4c}\frac{1}{s_{j_1j_2}^2}, \quad d_{\mathcal{M}\mathcal{M}}^{j_1j_2} = \frac{9}{32n^4c(c-1)}\frac{1}{s_{j_1j_2}^4}, \\
&d_{TJJ}^{j_1j_2j_3} = \frac{1}{16n^4}\Big(\frac{6}{c^2}\frac{1}{s_{j_1j_2}^2s_{j_1j_3}^2} - \frac{n^2-1}{c}\frac{1}{s_{j_2j_3}^2}\Big), \quad d_{JJ\mathcal{M}}^{j_1j_2j_3} = \frac{9}{16n^4c^2}\frac{1}{s_{j_1j_3}^2s_{j_2j_3}^2}, \\
&d_{JJJJ}^{j_1j_2j_3j_4} = \frac{9}{16n^4c^2}\Big(\frac{1}{s_{j_1j_2}^2s_{j_3j_4}^2} + \frac{1}{s_{j_1j_3}^2s_{j_2j_4}^2} + \frac{1}{s_{j_1j_4}^2s_{j_2j_3}^2}\Big), \quad d_{II(JJ)}^{j_1j_2} = -\frac{3}{160n^4c}\Big(\frac{15}{s_{j_1j_2}^4} - \frac{2(n^2+5)}{s_{j_1j_2}^2}\Big),
\end{aligned} \tag{2.12}$$

level	operators	?	degeneracy	number
1	J	$\times$	n	n
2	$\mathcal{M}$	$\times$	n	$\frac{n(n+1)}{2}$
	JJ	$\checkmark$	$\frac{n(n-1)}{2}$	
3	$\mathcal{K},\ \mathcal{O}$	$\times$	$2n$	$\frac{n(n^2+12n-1)}{6}$
	$TJ,\ J\mathcal{M}$	$\times$	$2n(n-1)$	
	JJJ	$\times$	$\frac{n(n-1)(n-2)}{6}$	
	$I(JJ)$	$\checkmark$	$\frac{n(n-1)}{2}$	
4	$\mathcal{L},\ \mathcal{N},\ \mathcal{P}_1,\ \mathcal{P}_2$	$\times$	$4n$	$\frac{n(n^3+26n^2+59n+10)}{24}$
	$T\mathcal{M},\ J\mathcal{O}$	$\times$	$2n(n-1)$	
	$J\mathcal{K},\ \mathcal{M}\mathcal{M}$	$\checkmark$	$\frac{3n(n-1)}{2}$	
	$TJJ,\ JJ\mathcal{M}$	$\checkmark$	$n(n-1)(n-2)$	
	$JJJJ$	$\checkmark$	$\frac{n(n-1)(n-2)(n-3)}{24}$	
	$I(TJ),\ I(J\mathcal{M})$	$\times$	$2n(n-1)$	
	$I(JJJ)$	$\times$	$\frac{n(n-1)(n-2)}{3}$	
	$II(JJ)$	$\checkmark$	$\frac{n(n-1)}{2}$	

Table 2: Extra holomorphic quasiprimary operators in SCFT^n after adding contributions of J. As in [31], we use the notations $I(JJ) = J\mathrm{i}\partial J - \mathrm{i}\partial JJ$, $II(JJ) = \partial J\partial J - \frac{1}{3}(J\partial^2 J + \partial^2 JJ)$, and the ones similar to them. In the 3rd column we use $\checkmark$ or $\times$ to denote whether the operators contribute to the Rényi mutual information or not. The 5th column agrees with the counting in (2.10).

with the definitions $s_{j_1j_2} \equiv \sin\frac{(j_1-j_2)\pi}{n}$, $c_{j_1j_2} \equiv \cos\frac{(j_1-j_2)\pi}{n}$, and the ones similar to them. Considering the relevant operators in table 1 and table 2, we get the Rényi mutual information organized by the large central charge

$$I_n^{(0,1)} = I_n^{\mathrm{L}} + I_{n,(0,1)}^{\mathrm{NL}} + \cdots . \tag{2.13}$$

Here the leading part I_n^{L} equals I_n^{cl} (2.1) and the next-to-leading part

$$I_{n,(0,1)}^{\mathrm{NL}} = I_{n,g}^{\text{1-loop}} + I_{n,v}^{\text{1-loop}}, \tag{2.14}$$

with $I_{n,g}^{\text{1-loop}}$ being (2.2) and $I_{n,v}^{\text{1-loop}}$ being (2.3).

2.4 Contributions of T, J, G and H in SCFT

By further adding the contributions of G and H, we have to consider the extra quasiprimary operators that are counted as

$$(1-x)\chi_{(0)}^n\chi_{(1)}^n(\chi_{(3/2)}^n - 1) = 2nx^{3/2} + 2n^2x^{5/2} + n(2n-1)x^3 + n^2(n+5)x^{7/2} + n(2n^2+n+1)x^4 + O(x^{9/2}), \tag{2.15}$$

with $\chi_{(0)}$ being (A.1), $\chi_{(1)}$ being (A.6) and $\chi_{(3/2)}$ being (A.15), and they are listed in table 3. For the relevant operators we have the normalization factors

$$\alpha_{GG} = \alpha_{HH} = -\frac{16c^2}{9}, \quad \alpha_{JGH} = \frac{16c^3}{27}, \quad \alpha_{I(GG)} = \alpha_{I(HH)} = -\frac{32c^2}{3}, \tag{2.16}$$

and the OPE coefficients

$$d_{GG}^{j_1j_2} = -d_{HH}^{j_1j_2} = -\frac{3\mathrm{i}}{32n^3c}\frac{1}{s_{j_1j_2}^3}, \quad d_{JGH}^{j_1j_2j_3} = -\frac{9}{64n^4c^2}\frac{1}{s_{j_1j_2}s_{j_1j_3}s_{j_2j_3}^2}, \quad d_{I(GG)}^{j_1j_2} = -d_{I(HH)}^{j_1j_2} = -\frac{3\mathrm{i}}{64n^4c}\frac{c_{j_1j_2}}{s_{j_1j_2}^4}. \tag{2.17}$$

level	operators	?	degeneracy	number
$\frac{3}{2}$	$G,\ H$	$\times$	$2n$	$2n$
$\frac{5}{2}$	$\mathcal{D}_1,\ \mathcal{D}_2$	$\times$	$2n$	$2n^2$
	$JG,\ JH$	$\times$	$2n(n-1)$	
3	$\mathcal{E}$	$\times$	n	$n(2n-1)$
	GH	$\times$	$n(n-1)$	
	$GG,\ HH$	$\checkmark$	$n(n-1)$	
$\frac{7}{2}$	$\mathcal{B},\ \mathcal{C},\ \mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3,\ \mathcal{F}_4$	$\times$	$6n$	$n^2(n+5)$
	$TG,\ TH,\ \mathcal{M}G$ $\mathcal{M}H,\ J\mathcal{D}_1,\ J\mathcal{D}_2$	$\times$	$6n(n-1)$	
	$JJG,\ JJH$	$\times$	$n(n-1)(n-2)$	
	$I(JG),\ I(JH)$	$\times$	$2n(n-1)$	
4	$\mathcal{I}_1,\ \mathcal{I}_2,\ \mathcal{I}_3,\ \mathcal{I}_4$	$\times$	$4n$	$n(2n^2+n+1)$
	$J\mathcal{E},\ G\mathcal{D}_1,\ G\mathcal{D}_2$ $H\mathcal{D}_1,\ H\mathcal{D}_2$	$\times$	$5n(n-1)$	
	JGH	$\checkmark$	$n(n-1)(n-2)$	
	$JGG,\ JHH$	$\times$	$n(n-1)(n-2)$	
	$I(GH)$	$\times$	$n(n-1)$	
	$I(GG),\ I(HH)$	$\checkmark$	$n(n-1)$	

Table 3: Extra holomorphic quasiprimary operators in SCFT^n after further adding contributions of G and H. The 5th column agrees with the counting in (2.15).

Considering the relevant operators in table 1, table 2, and table 3, we get the Rényi mutual information

$$I_n^{(0,1,3/2)} = I_n^{\rm L} + I_{n,(0,1,3/2)}^{\rm NL} + I_{n,(0,1,3/2)}^{\rm NNL} \cdots . \tag{2.18}$$

Here the leading part $I_n^{\rm L}$ equals $I_n^{\rm cl}$ (2.1), the next-to-leading part

$$I_{n,(0,1,3/2)}^{\rm NL} = I_{n,g}^{\text{1-loop}} + I_{n,v}^{\text{1-loop}} + I_{n,\psi}^{\text{1-loop}}, \tag{2.19}$$

with $I_{n,g}^{\text{1-loop}}$ being (2.2), $I_{n,v}^{\text{1-loop}}$ being (2.3), and $I_{n,v}^{\text{1-loop}}$ being (2.4), and the next-to-next-to-leading part is

$$I_{n,(0,1,3/2)}^{\rm NNL} = \frac{(n+1)(n^2-4)(n^4+40n^2+679)x^4}{302400cn^7} + O(x^5). \tag{2.20}$$

3 Rényi entropy of one interval on torus

In this section we investigate the Rényi entropy of one length ℓ interval on a torus with low temperature. The SCFT on torus is just the theory on a circle in thermal state. We expand the Rényi entropy by $q = \exp(-2\pi\beta/L)$, with β being the inverse temperature and L being the spatial period, to order q^2. In supergravity side we use the method in [15, 25], and in SCFT side we use the method in [24, 25].

3.1 Supergravity result

The classical part of holographic Rényi entropy is [15, 25]

$$S_n^{\rm cl} = \frac{c(n+1)}{6n}\log\Big(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\Big) - \Big(\frac{c(n-1)(n+1)^2}{9n^3}\sin^4\frac{\pi\ell}{L}\Big)q^2 + O(q^3). \tag{3.1}$$

The one-loop correction to holographic Rényi entropy from graviton is [15]

$$S_{n,g}^{\text{1-loop}} = -\frac{2n}{n-1}\bigg(\frac{1}{n^4}\frac{\sin^4\frac{\pi\ell}{L}}{\sin^4\frac{\pi\ell}{nL}} - 1\bigg)q^2 + O(q^3). \tag{3.2}$$

For the vector field we get

$$\begin{aligned} S_{n,v}^{\text{1-loop}} = -\frac{2n}{n-1}\bigg\{ & \bigg(\frac{1}{n^2}\frac{\sin^2\frac{\pi\ell}{L}}{\sin^2\frac{\pi\ell}{nL}} - 1\bigg)q + \bigg[\frac{2\sin^2\frac{\pi\ell}{L}}{n^4\sin^2\frac{\pi\ell}{nL}}\bigg(n^2\cos^2\frac{\pi\ell}{L} - n\sin\frac{2\pi\ell}{L}\cot\frac{\pi\ell}{nL} \\ & + \frac{\sin^2\frac{\pi\ell}{L}}{\sin^2\frac{\pi\ell}{nL}}\Big(\cos^2\frac{\pi\ell}{nL} + \frac{3}{4}\Big)\bigg) + \frac{\sin^4\frac{\pi\ell}{L}}{2n^4}\sum_{j=1}^{n-1}\bigg(\frac{1}{\big(\sin^2\frac{\pi j}{n} - \sin^2\frac{\pi\ell}{nL}\big)^2} + \frac{1}{\sin^4\frac{\pi j}{n}}\bigg) - \frac{3}{2}\bigg]q^2 + O(q^3)\bigg\}. \end{aligned} \tag{3.3}$$

For the two gravitinos we get

$$S_{n,\psi}^{\text{1-loop}} = -\frac{4n}{n-1}\bigg(\frac{1}{n^3}\frac{\sin^3\frac{\pi\ell}{L}}{\sin^3\frac{\pi\ell}{nL}} - 1\bigg)q^{3/2} + O(q^{5/2}). \tag{3.4}$$

3.2 Contributions of T in SCFT

In this subsection we consider the contributions of only the vacuum conformal family and repeat the calculation in [24, 25]. One has the density matrix in expansion of low temperature

$$\rho_{\text{vac}} = |0\rangle\langle 0| + \frac{q^2}{\alpha_T}|T\rangle\langle T| + O(q^3). \tag{3.5}$$

Denoting the interval by A and its complement by B, one gets the Rényi entropy

$$S_n^{(0)} = -\frac{1}{n-1}\log \text{tr}_A\big(\text{tr}_B|0\rangle\langle 0|\big)^n - \frac{2n}{n-1}\bigg(\frac{\text{tr}_A\big[\text{tr}_B|T\rangle\langle T|(\text{tr}_B|0\rangle\langle 0|)^{n-1}\big]}{\alpha_T\text{tr}_A\big(\text{tr}_B|0\rangle\langle 0|\big)^n} - 1\bigg)q^2 + O(q^3). \tag{3.6}$$

From the state operator correspondence one has

$$\frac{\text{tr}_A\big[\text{tr}_B|T\rangle\langle T|(\text{tr}_B|0\rangle\langle 0|)^{n-1}\big]}{\alpha_T\text{tr}_A\big(\text{tr}_B|0\rangle\langle 0|\big)^n} = \frac{\langle T(\infty)T(0)\rangle_{\mathcal{C}^n}}{\alpha_T}. \tag{3.7}$$

The n-fold complex plane $\mathcal{C}^n$ with coordinate z can be mapped to a complex plane $\mathcal{C}$ with coordinate f by the conformal transformation

$$f(z) = \bigg(\frac{z - \mathrm{e}^{\mathrm{i}\pi\ell/L}}{z - \mathrm{e}^{-\mathrm{i}\pi\ell/L}}\bigg)^{1/n}. \tag{3.8}$$

Then one gets

$$\frac{\langle T(\infty)T(0)\rangle_{\mathcal{C}^n}}{\alpha_T} = \frac{c(n^2-1)^2}{18n^4}\sin^4\frac{\pi\ell}{L} + \frac{1}{n^4}\frac{\sin^4\frac{\pi\ell}{L}}{\sin^4\frac{\pi\ell}{nL}}. \tag{3.9}$$

The Rényi entropy can be organized by large central charge as

$$S_n^{(0)} = S_{n,(0)}^{\text{L}} + S_{n,(0)}^{\text{NL}} + \cdots, \tag{3.10}$$

with the leading part $S_{n,(0)}^{\text{L}}$ equaling S_n^{cl} (3.1) and the next-to-leading part $S_{n,(0)}^{\text{NL}}$ equaling $S_{n,g}^{\text{1-loop}}$ (3.2).

3.3 Contributions of J in SCFT

Adding the contributions of J, we have to add the density matrix (3.5) by

$$\rho_J = \frac{q}{\alpha_J}|J\rangle\langle J| + \frac{q^2}{\alpha_{\mathcal{M}}}|\mathcal{M}\rangle\langle\mathcal{M}| + \frac{q^2}{\alpha_{\partial J}}|\partial J\rangle\langle\partial J| + O(q^3). \tag{3.11}$$

We get the extra contributions of J to the Rényi entropy

$$S_n^{(1)}=-\frac{2n}{n-1}\bigg\{\Big(\frac{\langle J(\infty)J(0)\rangle_{\mathcal{C}^n}}{\alpha_J}-1\Big)q+\bigg[\frac{\langle \mathcal{M}(\infty)\mathcal{M}(0)\rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{M}}}+\frac{\langle \partial J(\infty)\partial J(0)\rangle_{\mathcal{C}^n}}{\alpha_{\partial J}} \\ -\frac{n}{2}\Big(\frac{\langle J(\infty)J(0)\rangle_{\mathcal{C}^n}}{\alpha_J}\Big)^2+\frac{1}{2}\sum_{j=1}^{n-1}\frac{\langle J(\infty)J(0)J(\infty_j)J(0_j)\rangle_{\mathcal{C}^n}}{\alpha_J^2}-\frac{3}{2}\bigg]q^2+O(q^3)\bigg\}. \tag{3.12}$$

With some calculation we get

$$\frac{\langle J(\infty)J(0)\rangle_{\mathcal{C}^n}}{\alpha_J}=\frac{1}{n^2}\frac{\sin^2\frac{\pi\ell}{L}}{\sin^2\frac{\pi\ell}{nL}},\quad \frac{\langle \mathcal{M}(\infty)\mathcal{M}(0)\rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{M}}}=\frac{1}{n^4}\frac{\sin^4\frac{\pi\ell}{L}}{\sin^4\frac{\pi\ell}{nL}}, \\ \frac{\langle \partial J(\infty)\partial J(0)\rangle_{\mathcal{C}^n}}{\alpha_{\partial J}}=\frac{2}{n^4}\frac{\sin^2\frac{\pi\ell}{L}}{\sin^2\frac{\pi\ell}{nL}}\Big(n^2\cos^2\frac{\pi\ell}{L}-n\sin\frac{2\pi\ell}{L}\cot\frac{\pi\ell}{nL}+\frac{\sin^2\frac{\pi\ell}{L}}{\sin^2\frac{\pi\ell}{nL}}\Big(\cos^2\frac{\pi\ell}{nL}+\frac{1}{2}\Big)\Big), \\ \frac{\langle J(\infty)J(0)J(\infty_j)J(0_j)\rangle_{\mathcal{C}^n}}{\alpha_J^2}=\frac{\sin^4\frac{\pi\ell}{L}}{n^4}\bigg(\frac{1}{\big(\sin^2\frac{\pi j}{n}-\sin^2\frac{\pi\ell}{nL}\big)^2}+\frac{1}{\sin^4\frac{\pi j}{n}}+\frac{1}{\sin^4\frac{\pi\ell}{nL}}\bigg). \tag{3.13}$$

Then we reproduce the one-loop gravity result $S_{n,v}^{\text{1-loop}}$ (3.3).

3.4 Contributions of G and H in SCFT

Adding further the contributions of G and H, we need to add the density matrix (3.5) with (3.11) and

$$\rho_{G,H}=\frac{q^{3/2}}{\alpha_G}|G\rangle\langle G|+\frac{q^{3/2}}{\alpha_H}|H\rangle\langle H|+O(q^{5/2}). \tag{3.14}$$

We get the extra contribution for Rényi entropy from G and H

$$S_n^{(3/2)}=-\frac{2n}{n-1}\Big(\frac{\langle G(\infty)G(0)\rangle_{\mathcal{C}^n}}{\alpha_G}+\frac{\langle H(\infty)H(0)\rangle_{\mathcal{C}^n}}{\alpha_H}-2\Big)q^{3/2}+O(q^{5/2}). \tag{3.15}$$

With the correlation functions

$$\frac{\langle G(\infty)G(0)\rangle_{\mathcal{C}^n}}{\alpha_G}=\frac{\langle H(\infty)H(0)\rangle_{\mathcal{C}^n}}{\alpha_H}=\frac{1}{n^3}\frac{\sin^3\frac{\pi\ell}{L}}{\sin^3\frac{\pi\ell}{nL}}, \tag{3.16}$$

we reproduce the one-loop gravity result $S_{n,\psi}^{\text{1-loop}}$ (3.4).

4 Conclusion and discussion

In this paper, we have investigated the holographic Rényi entropy in the correspondence of $\mathcal{N}=2$ supergravity under AdS_3 background and two-dimensional $\mathcal{N}=(2,2)$ SCFT. For two short intervals on complex plane with small cross ratio $x\ll 1$ we got the Rényi mutual information to order x^4, and for one interval on torus with low temperature $q=\mathrm{e}^{-2\pi\beta/L}\ll 1$ we got the Rényi entropy to order q^2.

Such orders are lower than the previous results in literature, for examples, order x^8 for complex plane case in [20] and order q^4 for torus case in [28]. This is because in the present case we have the spin-1 field in supergravity side and the current with conformal weight 1 in SCFT side. The small spin in supergravity and small conformal weight in SCFT make the calculation to higher orders more complicated. To get higher order results in supergravity side we need to consider the contributions of m-CDW's for the complex plane case and the m-letter words in the torus case with $m\geq 3$. In SCFT side the number of degenerate quasiprimary operators would increase very quickly in higher levels, and the orthogonalization of these primary and quasiprimary operators would be very complicated. Although much more complicated, it would be nice if higher order results can be got in the future, and perhaps more effective method has to be developed.

Acknowledgements

The authors would like to thank Jun-Bao Wu and Hao Ouyang for valuable discussions and suggestions. And also we special thank Matthew Headrick for his Mathematica code *Virasoro.nb* which could be downloaded at the personal homepage `http://people.brandeis.edu/~headrick/Mathematica/index.html`. ZL was supported by NSFC Grant No. 11222549 and No. 11575202.

A Review of two-dimensional $\mathcal{N}$=(2,2) SCFT

In this appendix we review some useful properties of the $\mathcal{N}=(2,2)$ SCFT. More details can be found in [38].

In the $\mathcal{N}=(2,2)$ SCFT, we have the stress tensor $T(z)$, $\bar{T}(\bar{z})$, the current $J(z)$, $\bar{J}(\bar{z})$, and their supersymmetric partners $G(z)$, $\bar{G}(\bar{z})$, $H(z)$, $\bar{H}(\bar{z})$, as well as the operators that are constructed from them. These operators can be written as the products of the holomorphic ones and the anti-holomorphic ones, and there is one-to-one correspondence between the holomorphic and anti-holomorphic operators. So in this paper we only need to consider the holomorphic sector of the SCFT.

With only T, we get the character of the vacuum conformal family

$$\chi_{(0)}=\prod_{k=0}^{\infty}\frac{1}{1-x^{k+2}}. \tag{A.1}$$

The modes of T are denoted by L_m with $m\in \mathrm{Z}$, and they form the Virasoro algebra

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n}. \tag{A.2}$$

The primary operator of the vacuum conformal family is the identity operator. At level 2 we have T with normalization factor $\alpha_T=\frac{c}{2}$, and at level 4 we have quasiprimary operator

$$\mathcal{A}=(TT)-\frac{3}{10}\partial^2T,\quad \alpha_{\mathcal{A}}=\frac{c(5c+22)}{10}. \tag{A.3}$$

Here we use the bracket '()' to denote normal ordering. Under a general conformal transformation $z\to f(z)$, they transform as

$$T(z)=f'^2T(f)+\frac{c}{12}s,\quad \mathcal{A}(z)=f'^4\mathcal{A}(f)+\frac{5c+22}{30}s\Big(f'^2T(f)+\frac{c}{24}s\Big), \tag{A.4}$$

with the Schwarzian derivative being

$$s(z)=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2. \tag{A.5}$$

By adding the current operator J with conformal weights $(1,0)$ we get the bosonic part of the SCFT[1], and we have to multiply the character (A.1) by

$$\chi_{(1)}=\prod_{k=0}^{\infty}\frac{1}{1-x^{k+1}}. \tag{A.6}$$

The Virasoro algebra (A.2) is supplemented by

$$[L_m,J_n]=-nJ_{m+n},\quad [J_m,J_n]=\frac{c}{3}m\delta_{m+n}. \tag{A.7}$$

[1]The term 'bosonic part' is not so precise, since with the fermionic operators G and H we can also construct bosonic operators. Here by bosonic part we mean the sector with operators that are constructed solely by T and J, without G or H.

level	1	2	3	4
J	J		$\mathcal{K}$	$\mathcal{L}$
$\mathcal{M}$		$\mathcal{M}$		$\mathcal{N}$
$\mathcal{O}$			$\mathcal{O}$	
$\mathcal{P}_1, \mathcal{P}_2$				$\mathcal{P}_1, \mathcal{P}_2$

Table 4: Up to level 4 the extra quasiprimary operators because of the adding of J. In the first column we list the primary operator of each conformal family.

Note that the algebra of SCFT bosonic part is close and self-consistent. Besides the ones in vacuum conformal block, there are other quasiprimary operators that can be counted as

$$(1-x)\chi_{(0)}(\chi_{(1)}-1) = x + x^2 + 2x^3 + 4x^4 + O(x^5). \tag{A.8}$$

Among these quasiprimary operators, the primary ones can be counted as

$$\frac{\chi_{(0)}(\chi_{(1)}-1)}{\chi} = x + x^2 + x^3 + 2x^4 + O(x^5), \quad \chi \equiv \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{A.9}$$

These primary and quasiprimary operators are listed in table 4. For the primary operator J we have the normalization $\alpha_J = \frac{c}{3}$, for quasiprimary operator $\mathcal{K}$ we have

$$\mathcal{K} = (TJ) - \frac{1}{2}\partial^2 J, \quad \alpha_{\mathcal{K}} = \frac{c(c+2)}{6}, \tag{A.10}$$

and for primary operator $\mathcal{M}$ we have

$$\mathcal{M} = (JJ) - \frac{2}{3}T, \quad \alpha_{\mathcal{M}} = \frac{2c(c-1)}{9}. \tag{A.11}$$

Under a conformal transformation $z \to f(z)$ we have

$$J(z) = f' J(f), \quad \mathcal{K}(z) = f'^3 \mathcal{K}(f) + \frac{c+2}{12} s f' J(f). \tag{A.12}$$

In this paper we need the structure constants

$$C_{JJJ} = 0, \quad C_{TJJ} = \frac{c}{3}, \quad C_{JJ\mathcal{M}} = \frac{2c(c-1)}{9}, \tag{A.13}$$

as well as the four-point function on complex plane

$$\langle J(z_1)J(z_2)J(z_3)J(z_4)\rangle_{\mathcal{C}} = \frac{c^2}{9}\Big(\frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{13}^2 z_{24}^2} + \frac{1}{z_{14}^2 z_{23}^2}\Big), \tag{A.14}$$

with the shorthand $z_{jk} = z_j - z_k$.

By further adding the fermionic operators G and H with conformal weights $(3/2, 0)$, we get the SCFT and the character is the product of (A.1), (A.6) and

$$\chi_{(3/2)} = \prod_{k=0}^{\infty} \big(1 + x^{k+3/2}\big)^2. \tag{A.15}$$

Note that there are two primary operators at level 3/2 and so there is a power 2 in the right-hand side of the above equation. As argued in [30], in large central charge limit we only need to consider the Neveu-Schwarz

level	3/2	5/2	3	7/2	4
G	G			$\mathcal{B}$	
H	H			$\mathcal{C}$	
$\mathcal{D}_1, \mathcal{D}_2$		$\mathcal{D}_1, \mathcal{D}_2$			
$\mathcal{E}$			$\mathcal{E}$		
$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$				$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$	
$\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$					$\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$

Table 5: The extra primary and quasiprimary operators because of the adding of G and H.

sector for the fermionic operators. We have the modes G_r, H_s with $r, s \in \mathrm{Z} + 1/2$, and to compose the $\mathcal{N} = 2$ super Virasoro algebra we need (A.2), (A.7), and

$$\begin{aligned}
&[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}, \quad [L_m, H_r] = (\frac{m}{2} - r)H_{m+r}, \\
&[J_m, G_r] = H_{m+r}, \quad [J_m, H_r] = G_{m+r}, \quad \{G_r, H_s\} = -2(r-s)J_{r+s}, \\
&\{G_r, G_s\} = 4L_{r+s} + \frac{2c}{3}(r^2 - \frac{1}{4})\delta_{r+s}, \quad \{H_r, H_s\} = -4L_{r+s} - \frac{2c}{3}(r^2 - \frac{1}{4})\delta_{r+s}.
\end{aligned} \tag{A.16}$$

The extra quasiprimary operators can be counted as

$$(1-x)\chi_{(0)}\chi_{(1)}(\chi_{(3/2)} - 1) = 2x^{3/2} + 2x^{5/2} + x^3 + 6x^{7/2} + 4x^4 + O(x^{9/2}), \tag{A.17}$$

among which the primary ones can be counted as

$$\frac{\chi_{(0)}\chi_{(1)}(\chi_{(3/2)} - 1)}{\chi} = 2x^{3/2} + 2x^{5/2} + x^3 + 4x^{7/2} + 4x^4 + O(x^{9/2}). \tag{A.18}$$

These operators are listed in table 5. For G and H we have normalization constants $\alpha_G = -\alpha_H = \frac{4c}{3}$, and under conformal transformation they transform as

$$G(z) = f'^{\frac{3}{2}} G(f), \quad H(z) = f'^{\frac{3}{2}} H(f). \tag{A.19}$$

There are useful structure constants

$$C_{JJG} = C_{JJH} = C_{JGG} = C_{JHH} = 0, \quad C_{JGH} = -\frac{4c}{3}. \tag{A.20}$$

B Some useful summation formulas

In this appendix we collect some summation formulas that are useful to calculation in section 2. We define

$$f_m = \sum_{j=1}^{n-1} \frac{1}{\left(\sin \frac{\pi j}{n}\right)^{2m}}, \tag{B.1}$$

and we need

$$\begin{aligned}
&f_2 = \frac{(n^2-1)(n^2+11)}{45}, \quad f_3 = \frac{(n^2-1)(2n^4+23n^2+191)}{945}, \\
&f_4 = \frac{(n^2-1)(n^2+11)(3n^4+10n^2+227)}{14175}.
\end{aligned} \tag{B.2}$$

We have

$$\sum_{\neq} \frac{1}{s^4_{j_1j_2} s^4_{j_1j_3}} = \frac{4n(n^2-1)(n^2-4)(n^2+11)(n^2+19)}{14175},$$
$$\sum_{\neq} \frac{1}{s^2_{j_1j_2} s^2_{j_1j_3} s^2_{j_2j_3}} = \frac{2n(n^2-1)(n^2-4)(n^2+47)}{945}, \tag{B.3}$$
$$\sum_{\neq} \frac{1}{s^2_{j_1j_2} s^2_{j_1j_3} s^4_{j_2j_3}} = \frac{2n(n^2-1)(n^2-4)(n^4+40n^2+679)}{14175},$$

with definition $s_{j_1j_2} \equiv \sin\frac{\pi(j_1-j_2)}{n}$ and the ones similar to it, and the summation is over $0 \leq j_{1,2,3} \leq n-1$ with constraints $j_1 \neq j_2$, $j_1 \neq j_3$, $j_2 \neq j_3$. We have

$$\sum_{\neq} \frac{1}{s^4_{j_1j_2} s^4_{j_3j_4}} = \frac{n(n^2-1)(n-2)(n-3)(n^2+11)(7n^3+13n^2+93n+127)}{14175}, \tag{B.4}$$
$$\sum_{\neq} \frac{1}{s^2_{j_1j_2} s^2_{j_3j_4} s^2_{j_1j_3} s^2_{j_2j_4}} = \frac{4n(n^2-1)(n^2-4)(n^2-9)(n^2+119)}{14175},$$

with the summation being over $0 \leq j_{1,2,3,4} \leq n-1$ and the constraints being $j_1 \neq j_2$, $j_1 \neq j_3$, $j_1 \neq j_4$, $j_2 \neq j_3$, $j_2 \neq j_4$, $j_3 \neq j_4$. Finally we have

$$\sum{}' \frac{1}{s^2_{j_1j_2} s^2_{j_2j_3} s^2_{j_3j_4} s^2_{j_4j_1}} = \frac{4n(n^2-1)(n^2-4)(3n^4+170n^2-653)}{14175}, \tag{B.5}$$

with the summation being over $0 \leq j_{1,2,3,4} \leq n-1$ and the constraints being

$$j_1 \neq j_2, \;\; j_2 \neq j_3, \;\; j_3 \neq j_4, \;\; j_4 \neq j_1, \;\; (j_1, j_2) \neq (j_3, j_4). \tag{B.6}$$

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